Center and Centroid of graphson Dominating Sets

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Abstract-Let S be a minimum dominating set of aconnected undirectedgraphG = (V, E). Inthispaper we study and analyzecenterandcentroidofgraphsondominatingsets. The$S$-distanceofavertex$u$isdefinedby$d_S(u)=\sum_{v\in S} d(u,v)$.Avertex$uofG$issaidtobea$S$-centroidvertexifitminimize$d_S(u)=\sum_{v\in S} d(u,v)$.The$S$-centroidgraphof G is the graph induced by the set of all S-centroid vertices and is denoted by $C_{\gamma}(G)$. The S -eccentricity of a vertex u is definedby$e_S(u)=max\{d(u,v):v\in S\}$.From this wecandefinethe$S$-radiusof$G$as$rad_S(G)=min\{e_S(u):u\in V(G)\}$andthe$S$-diameterof$Gasdiam_S(G)=\max\{e_S(u):u\in V(G)\}$.Nowthe vertexvisa$S$-centralvertexif$e_S(u)=rad_S(G)$, and the S-center $C_S(G)$ is the graph induced by the set of all S-central vertices. An edge e of G is said to be a S-centroidedgeifitminimize$d_S(e)=\sum_{v\in S} d(e,v)$.The$S$-centroidgraphof$G$withrespecttoedge-to-vertexdistanceis the graph induced by the setofall$S$-centroid edges and is denoted by$C_{2_{\gamma}}(G)$. The S-eccentricityof an edge $e$is defined by$e_{2_S}(e)=max\{d(e,v):v\in S\}$.Fromthiswecandefinethe$S$-radiusof$Gasr_{2_S}(G)=min\{e_{2_S}(e):e\in E(G)\}$andthe S-diameterof$Gasd_{2_S}(G)=\max\{e_{2_S}(e):e\in E(G)\}$.Nowthevertexvisa$S$-centraledgeif$e_{2_S}(e)=rad_S(G)$,andthe $S-center$$C_{2_S}(G)$withrespecttoedge-to-vertexdistanceisthegraphinducedby thesetofall S-centraledges.

Keywords– S-status, S-centroid, S-center, graphs, dominating set, eccentricities, centre

INTRODUCTION

Centrality is one of the fundamental notions which have established close connectionbetween graph theoryandvariousotherareas like socialnetworks.The mainobjectiveofanyfacilitylocationproblemis toidentify thelocationforthe facilityfor communityora setofcustomers suchthat the distancebetweenthelocationandthe community or customers is minimized.

The graph center problem is interesting from both a structural and the algorithmic point of view.P, J. Slater generalized the concept of center of a graph to center of a profile of the vertex set of the graph. Graphs are often used to model such things as street networks and communication networks, and many mathematical problems have arisen from different instances of the question of what is an optimal location for a facility in a graph. In the almost cases the type of facility to be established is one for which a “central” location is optimal. The study of minimizing the total distance from the facility to certain important sites is the concept, S-distance.

Weconsideronly finitesimpleundirectedconnectedgraphs.Foragraph G,Vdenotesthevertexsetand$E$ denotestheedgeset.

In 1964, Hakimi [4] considered the facility location problems as vertex-to-vertex distance in graphs. Foranytwo vertices$u$ and $v$inaconnected graph G, thedistance $d(u,v)$ isthelengthofashortest $u-v$pathin G. For a vertex v in G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of $G$is its radius and the maximum eccentricity is its diameter, denoted by $rad(G)$ and $diam(G)$ respectively. A vertex v in G is a central vertex if $e(v)=rad(G)$ and the subgraph induced by the central vertices of G is the center $Cen(G)$ or $C(G)$ of G.

In2010,Santhakumaran[7] introduced the facilitylocation problemasedge-to-vertexdistance in graphsas follows:Foranedge$e$andavertex$v$inaconnectedgraph G,theedge-to-vertexdistanceisdefinedby$d(e,v)=min\{d(u,v):uisanyendvertexofe\}$.Theedge-to-vertexeccentricityisdefinedby$e_2(e)=max\{d(e,v):v\in V\}$.Avertex$vofG$suchthat$e_2(e)=d(e,v)$iscalledanedge-to-vertexeccentricvertexof$e$.Theedge-tovertexradius$r_2$of Gisdefinedby$r_2=min\{e_2(e):e\in E\}$andtheedge-to-vertexdiameter$d_2$ofGisdefined by $d_2=max\{e_2(e):e\in E\}$. An edge $e$for which $e_2(e)$is minimum is called an edge-to-vertex central edge of $G$and the set of all edge-to- vertex central edges of $G$is the edge-to-vertex center $C_2(G)$of G. For a set $S$of vertices, let the closed interval $I[S]$ of S be the union of the closed intervals $I[v,w]$ over all the pairs of vertices v and w in S. For a set S of edges, let the closed interval $I_2[S]$ of S be the union of the closed intervals $I[e,f]$ over all the pairs of edges e and $f$whose end points belongs to V. A set $S\subseteq V(G)$ is a dominating set of$G$ if every vertex in $V(G)$ is either belongs to S or is adjacent to a vertex of S. The minimum dominating set or $\gamma -$ set is a dominating set of minimum cardinality.

Inthispaperweconsidertheset$S$astheminimumdominatingset.

I. S-CENTERANDS-CENTROIDON γ−SETSWITHRESPECTTOVERTEX-TO-VERTEXDISTANCE

In this section we construct some basic theorems on S-Center and S-Centroid on minimum dominating sets with respect to vertex- to- vertex distance. Let $G=(V,E)$ be a graph and Sbe any subset of the vertex set V. Then the S-distance of a vertex $u$is defined by $d_S(u)=\sum_{v\epsilon S} d(u,v)$. A vertex $u$of $G$is said to be aS-centroid vertex if it minimize $d_S(u)= \sum_{v\epsilon S} d(u,v)$. The S-centroid graph of $G$is the graph induced by the set of all S-centroid vertices and is denoted by $C_\gamma(G)$.

The$S$-eccentricityofavertex $u$isdefined by$e_S(u)=\max\{d(u,v):v\in S\}$.Fromthiswecan definethe S-radius of$G$as$rad_S(G)=\min\{e_S(u):u\in V(G)\}$andthe$S$-diameterof Gas $diam_S(G)=\max\{e_S(u):u\in V(G)\}$. Nowthe vertex v is a S-central vertex if $e_S(u) = rad_S(G)$, and the S-center $C_S(G)$ is the graph induced by the set ofall S-central vertices. Inthe followinggraph, we find S-centroid and S-center with respect to vertex-to- vertex distance.

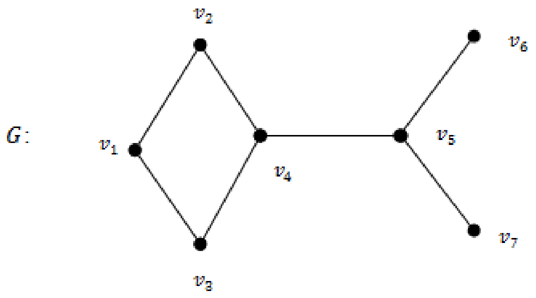

Figure1.Agraph$G$

Inthisgraph,theminimumdominatingsetis$S=\{v_1,v_5\}$.Here$d_S(v_1)=3$,$d_S(v_2)=3$,$d_S(v_3)=3$,$d_S(v_4)=3$,$d_S(v_5)=3$,$d_S(v_6)=5$,$d_S(v_7)=5$andalsothesetofall$S$-centroidverticesis$\{v_1,v_2,v_3,v_4,v_5\}$.Here, $e_S(v_1)=\max\{d(v_1,v_1),d(v_1,v_5)\}=3$, $e_S(v_2)=\max\{d(v_2,v_1),d(v_2,v_5)\}=2$, $e_S(v_3)= \max\{d(v_3, v_1), d(v_3,v_5)\}=2$, $e_S(v_4)=\max\{d(v_4, v_1), d(v_4, v_5)\}=2$and so on.

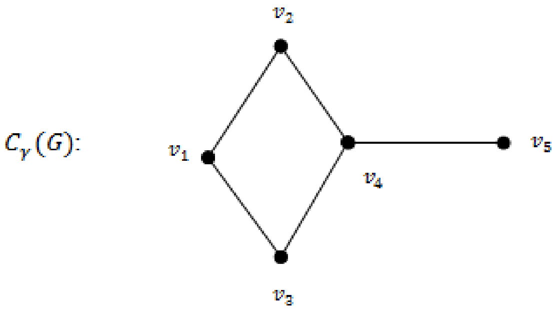

Figure2.The$S$–centroidofthe graphshowninFigure1

Hence $rad_S(G) = 2$, and so the set ofall S-central verticesis$\{v_2,v_3,v_4\}$. Fromfigure 1 we canconstruct the$S$– centroid and S-center.

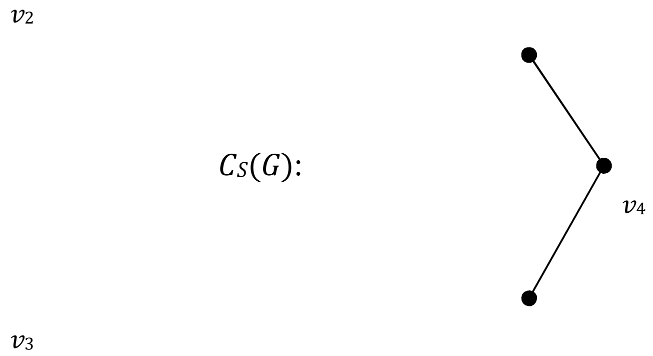

Figure 3. The S–center of the graph G represented in Figure 1

Theorem 2.1

Let S be a set of a connected graph G. Then $rad_S(G) \leq rad(G)$.

Proof:

Let $V(G)=\{v_1,v_2,\ldots,v_n\}$ be the vertex set of G and let $S=\{v_1,v_2,\ldots,v_k\}$ be a $\gamma-$set of $G,(k\leq n)$. Now $rad_S(G)=\min\{e_S(v_1),e_S(v_2),e_S(v_3),\ldots,e_S(v_n),v_i \in V(G),i=1 to n\}$. We have to prove $rad_S(G)\leq rad(G)$. Let $rad(G)=m$ and let u be a central vertex of G. Then $e(u)=d(u,w)=m$ where w is the eccentric vertex of u. Now we analyze the location of the vertex w in two cases.

case(i): Suppose $w\in S$, then clearly $rad_S(G)=rad(G)$.

case(ii): Suppose $w \notin S$, then S contains the vertices v_i such that $d(u,v_i)<m \forall i=1 to m$.

Therefore, $rad_S(G)\leq rad(G)$. ■

Theorem 2.2

Let S be a set of a connected graph G. Then $diam_S(G)\leq diam(G)$.

Remark 2.3

If $S=\{v\}$, then $rad_S(G)=0$ and $C_S(G)=\{v\}$. From this we conclude the property that $rad_S(G) \leq diam_S(G)\leq 2rad_S(G)$ is not hold.

Theorem 2.4

Let S be a minimum dominating set of a connected graph G. Then the S–center is a subset of $I[S]$.

Proof:

Let S be a minimum dominating set of a connected graph G, say $S=\{u_1,u_2,\ldots,u_k\}$. Let v be a S-central vertex. Then we have to show that v belongs to $I[S]$. Suppose to the contrary that we take v is not in $I[S]$. We have to show that v is not a S-central vertex. Then v is dominated by a vertex $u_i(1 \leq i \leq k)$ of S. Then $e_S(v) = \max\{d(v,u_i):u_i\in S\}$ is not minimum. Therefore v is not a S –central vertex. Hence the set of all S –central vertices is a subset of $I[S]$. ■

Corollary 2.5

The S–center $C_S(G)$ is contained in the Steiner tree of S.

Remark 2.6

Let S be a minimum dominating set of a connected graph G. Then the central vertices and the set of all S- central vertices need not be same.

Theorem 2.7

The S-center of any connected graph G lies within a block.

Proof:

Suppose the S-center $C_s(G)$ of a connected graph lies in more than one block. Then G contains a cut vertex v such that G-v has components G_1 and G_2, each of which contains a S-central vertex of G. Let u be an S-eccentric vertex of v and let P be a u-v path of length $e_s(v)$. Then v contains no vertex from at least one of G_1 and G_2, say G_1. Let w be a S-central vertex of G_1 and P' be a w-v geodesic in G. Suppose u has another S-eccentric vertex x, then $e_s(u)= d(u,w)=m$ and $e_s(u)=d(u,x)=m$. Assume that $w\in S$ and $x \notin S$ then $rad_S(G)=rad(G)$.

Then $e_s(w)\geq d(w,v) + d(v,u) \geq 1 + e_s(v)$ which is a contradiction to w is a S-central vertex. Then all the S- central vertices must lie in a single block.

Theorem 2.8

Proof: Inatree$T$,every$S$-centralvertexisacentralvertex.

Let$u$be aS-centralvertex. Wehave is,e(u)Grad(T)⇒$e(u)>rad(T)$.Wetake$e(u)=rad(T) + 1$.Thereforethereexistsaeccentricvertex$v\in V$ for$u$,$e(u) = d(u,v) = rad(T) +1$.Since$uisaS$-centralvertex,$e_S(u) = rad_S(T)$,thereexistsa$S$-eccentric vertex w∈$S$for u. That is, $d(u,w)=e_S(u)=rad_S(T)$. Hence $v\in S$or vøS.
case(i): Supposethat $v\in S$.Since$v$is a farthestvertexto$u$in$V$,$v$isthefarthest vertexto$u$in$S$$d(u,v)=e_S(u)=$ $rad_S(T)$.$rad(T)+1=e_S(u)=rad_S(T)$.Hence$rad(T)$ $+1=rad_S(T)$whichimplies$rad(T)<rad_S(T)$ whichisacontradictionto$u$isa$S$-centralvertex.Hence$u$isacentralvertex.
case(ii):Suppose$v$øS.Since$S$isa$\gamma-$set,$v$isdominatedby
Proof:

u. Since u is a S-central vertex, it is clear that $v$is an end vertex. Since S is a $\gamma -$ set, v is dominated by at least one vertex,say$x\in S$.Hencewehave$d(v,x)=1$.Since$v$isthefarthestvertexto$u$,$x$liesbetween$u$and$v$.Thatis, $d(u,v)=d(u,x)+d(x,v)$whichimpliesthat$d(u,x)=rad_S(T)$.Therefore,$rad(T)=rad_S(T)+1$.Hence, $rad_S(T)=rad(T)-1$.
Theorem2.10

Inatree$T$,everycentralvertexisa$S$-centralvertex.

Proof:

Let$u$beacentralvertex.Now$u$isacentralvertex,thereexistsa$S$-eccentricvertex$v$in$V$suchthat $d(u,v)=e(u)=rad(T)$. We have to show that u is S-central vertex. Hence there are two cases, $v\in S$or
■

toshowthat$u$iscentralvertex.Suppose $u$isnota centralvertex.That

atleastonevertexsay$x\in S$.Hencewehave $d(v,x)=1$.Since$v$isthefarthestvertexto$u$,$x$liesbetween $u$and$v$.i.e.,$d(u,v)=d(u,x) + d(x,v)$which implies$rad(T) + 1=d(u,x) + d(x,v)$.Therefore,$d(u,x)=rad(T)$.Since$x\in S$,$x$isthefarthestvertexto$u$in S, $d(u,x) =rad_S(T)$. Thereforewe get,$rad_S(T) = rad(T)$whichis a contradiction to u is aS-centralvertex. In both the cases on v there are contradictions on u is a S-central vertex. Hence u is a central vertex.

Fortree$T$,$rad_S(T)=rad(T)-1$if Scontains no endvertices.

Let$u$beacentralvertexin$T$.Itimpliesthat$rad(T)=d(u,v)=e(u)$,where$v$isaneccentricvertexfor
vøS.
case(i):Suppose$v\in S$.Since$v$isafarthestvertexto$u$,$v$isthefarthestvertexto$u$in$S$.$d(u,v)=e_S(u)=$ $rad_S(T)$.Hence$u$ isa$S$-centralvertex.
case(ii):Suppose$v$øS.Since$S$isa$\gamma-$set,$v$isdominatedby atleastonevertexsay,$x\in S$.Hence$d(v,x)=1$ andso$x$isthefarthestvertexto$u$andisin$S$andso$d(u,x)=e_S(u)$.Since$v$isthefarthestvertexto$u$and$v$øS, ■
$d(u,x) = e_S(u)$. Therefore, $d(u,x) = rad(T)$. Since $x\in$ S, x is the farthest vertex to u in S, $d(u,x) = rad_S(T)$. Therefore, $rad_S(T) = rad(T)$, which is a contradiction to u a S-central vertex. In both the cases there are contradictions on u a S-central vertex and so u is a central vertex.

II. S–CENTERAND S–CENTROIDON$\gamma-$SETSWITHRESPECTTOEDGE-TO-VERTEXDISTANCE

Inthissectionweconstructsomebasictheoremson$S$-Centerand$S$-Centroidonminimumdominating sets with respect to edge - to- vertex distance. The S-distance of a edge e is defined by$d_S(e) = \sum_{v\epsilon S} d(e,v)$. An edge e of $G$is said to be a S-centroid edge if it minimize $d_S(e)=\sum_{v\epsilon S} d(e,v)$.

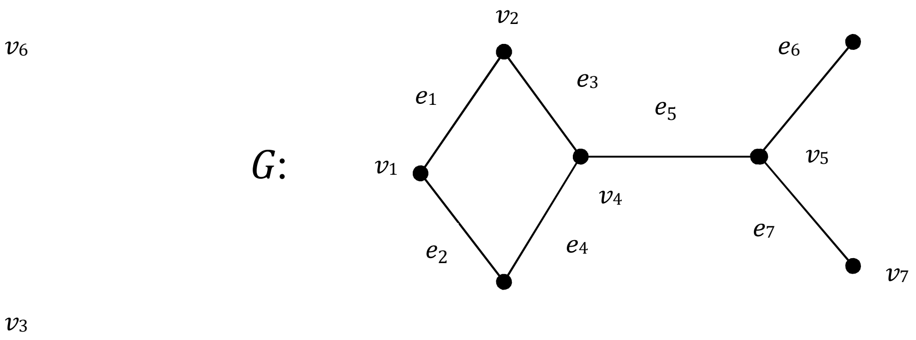

Figure3.Agraph

The S-centroid graph of $G$with respect to edge - to- vertex distance is the graph induced by the set of all S-centroid edgesandisdenotedby$C_{2_\gamma}(G)$.The$S$-eccentricityofanedge$e$isdefinedby$e_{2_S}(e)=\max\{d(e,v):v\in S\}$.From this we can define the S-radius of G as$r_{2_S}(G) = \min\{e_{2_S}(e){:}e \in E(G)\}$ and the S-diameter of G as $d_{2_S}(G) = \max\{e_{2_S}(e){:}e \in E(G)\}$. Now the vertex v is a S-central edge if $e_{2_S}(e) = rad_S(G)$, and the S-center $C_{2_S}(G)$with respect to edge - to- vertex distance is the graph induced by the set of all S-central edges.

Intheabovegraph,wefind$S$-centroidand$S$-centerwithrespecttoedge-to-vertexdistance.

Letusassume$e_1 = v_1v_2$,$e_2 = v_1v_3$,$e_3 = v_2v_4$,$e_4 = v_3v_4$,$e_5 = v_4v_5$,$e_6 = v_5v_6$,$e_7 = v_5v_7$.Inthisgraphthe minimumdominatingsetis$S$=$\{v_1,v_5\}$.Here$d_S(e_1)=0 + 2=2$,$d_S(e_2)=0 + 2=2$,$d_S(e_3)=1 + 1= 2$,$d_S(e_4) = 1 + 1 = 2$,$d_S(e_5) = 2 + 0 = 2$,$d_S(e_6) = 3 + 0 = 3$,$d_S(e_7) = 3 + 0 = 3$andalsothesetofall$S$- centroid edges is $\{e_1, e_2, e_3, e_4, e_5\}$.

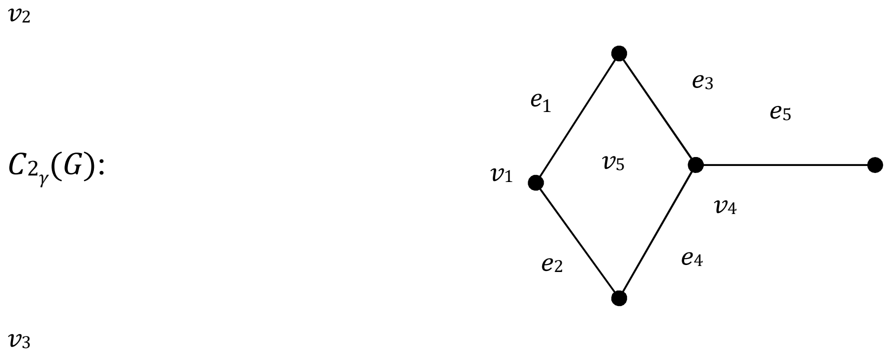

Figure4.The$S$–centroid withrespectedge-to-vertexdistanceofthegraphGrepresentedinFigure3

Here,$e_{2_S}(e_1)=\max\{d(e_1,v_1),d(e_1,v_5)\}=2$,$e_{2_S}(e_2)=\max\{d(e_2,v_1),d(e_2,v_5)\}=2$,
$e_{2_S}(e_3)=\max\{d(e_3,v_1),d(e_3,v_5)\}=2$,$e_{2_S}(e_4)=\max\{d(e_4,v_1),d(e_4,v_5)\}=1$,
$e_{2_S}(e_5)=\max\{d(e_5,v_1),d(e_5,v_5)\}=3$,$e_{2_S}(e_6)=\max\{d(e_6,v_1),d(e_6,v_5)\}=3$,
$e_{2_S}(e_7)=\max\{d(e_7,v_1),d(e_7,v_5)\}=3$.Hence,$r_{2_S}(G)=1$,andsotheS-centeris$C_{2_S}(G)=\{e_4\}$.

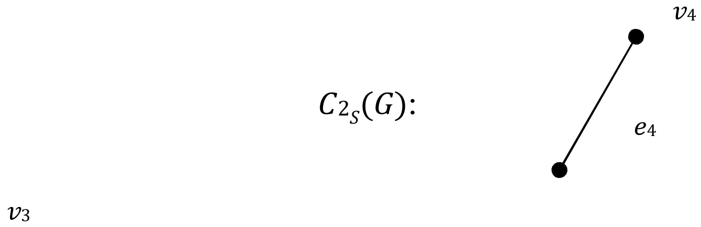

Figure5.The$S$–centerwithrespectedge-to-vertexdistanceofthegraphGrepresentedinFigure3

We knowthat the S -eccentricityofa edge e isdefined by$e_{2_S}(e) = \max\{d(e,v){:}v\in S\}$. Fromthis we candefine the S-radius as $r_{2_S}(G)=\min\{e_{2_S}(e){:}e\in E(G)\}$and the S-diameter as $d_{2_S}(G)=\max\{e_{2_S}(e){:}e\in E(G)\}$.
Now$v$is aS-centraledgeif$e_{2_S}(e)=r_{2_S}(G)$andthe S-center$C_{2_S}(G)$is thesetof all$S$-central edges.
In the Figure 2, for the $\gamma -$ Set S =$\{v_1,v_5\}$, $r_{2_S}(G) = 1$ and $d_{2_S}(G) = 3$.

Theorem 3.1

Let Sbeasetofaconnected graphG. Then$r_{2_S}(G)\leq r_2(G)$.

Proof.

Let$E(G)=\{e_1,e_2,\ldots,e_n\}$bethevertex setof Gandlet$S=\{v_1,v_2,\ldots,v_k\}$beaγ–setofG,$(k\leq n)$.
Now$r_{2_S}(G)=\min\{e_{2_S}(e_1),e_{2_S}(e_2),e_{2_S}(e_3),\ldots,e_{2_S}(e_n),e_i\in E(G),i=1 to n\}$.Wehavetoprovethat$r_{2_S}(G)\leq r_2(G)$.Let $r_2(G)=m$andlet$e$beacentralvertexofG.Then$e_2(e)=d(e,w)=m$where wistheeccentricvertex of$e$.Nowweanalyzethelocationofthe vertexwintwo cases.

case(i).Supposethatw$\in S$,thenclearly$r_{2_S}(G)=r_2(G)$.
case(ii).Supposethatw$\notin S$,then$S$containstheverticesv$_i$suchthat$d(e,v_i)<m$foreveryi$=1$ to m.Hence $r_{2_S}(G)<r_2(G)$.Therefore,$r_{2_S}(G)\leq r_2(G)$. ■

Theorem3.2

Let$S$bea set of a connectedgraph G.Then$d_{2_S}(G) \leq d_2(G)$.

Proof.

TheproofissimilarastheproofofTheorem2.3. ■

Remark3.3

Let Sbe aminimum dominatingset of a connectedgraph$G$. Then the central edges andthe set of all S- central edges need not be same.

Theorem3.4

InatreeT,every$S$-centraledgeisacentraledge.

Proof.

Let$e$beaS-centraledge.Wehavetoshowthat$e$iscentraledge.Supposeeisnotacentraledge.i.e., $e_{2_S}(e)$G$r_2(T) \Rightarrow e_{2_S}(e) > r_2(T)$.Wetake$e_{2_S}(e) = r_2(T)+1$.Thereforethereexistsaeccentricvertex$v \in V$for e,$e_{2_S}(e) = d(e,v) = r_2(T)+1$.Since$e$isaS-centraledge,$e_{2_S}(e) = r_{2_S}(T)$,thereexistsa$S$-eccentricvertex w∈$Sfore$.Thatis,$d(e,w) = e_{2_S}(e) = r_{2_S}(T)$.Hence$v \in Sorv ø S$.

Case(i):Suppose$v \in S$.Since$v$isafarthestvertextoe in$V$,$v$isthefarthestvertextoeinS,$d(e,v) = e_{2_S}(e) = r_{2_S}(T)$.$r_2(T) + 1 = e_{2_S}(e) = r_{2_S}(T)$.Hence$r_2(T) + 1 = r_{2_S}(T)$whichimplies$r_2(T) < r_{2_S}(T)$whichisa contradiction to e is a S-central edge. Hence e is a central edge.

Case(ii):Suppose$v ø S$.SinceSisa$\gamma -$set,visdominatedbyatleastonevertexsay$x \in S$.Hencewehave $d(v,x) = 1$.Since$v$isthefarthestvertexto$e$,$x$liesbetween$e$and$v$.Thatis,$d(e,v) = d(e,x) + d(x,v)$which implies$r_2(T)+1 = d(e,x) + d(x,v)$.Therefore,$d(e,x) = r_2(T)$.Since$x \in S$,$x$isthefarthestvertexto$einS$, $d(e,x) = r_{2_S}(T)$. Therefore we get, $r_{2_S}(T) = r_2(T)$which is a contradiction to e, $e$is a S-central edge. In both the cases on v there are contradictions on e is a S-central edge. Hence, e is a central edge. ■

CONCLUSION

Centre and centroid are two crucial emergency locations that are examined in this research. In contrast to the usual practice of defining them using vertices, they are defined here by fixing a preponderant set of graphs. Additionally, they are described in relation to edges. In addition, we define vertex to edge and edges to edge lengths and verify and list their qualities.

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